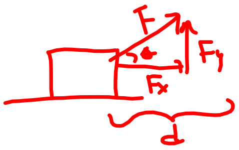
# Torque and Moment of Inertia

## Torque ($\vec{\tau}$)




$$\vec{\tau} = \vec{r} \times \vec{F}$$

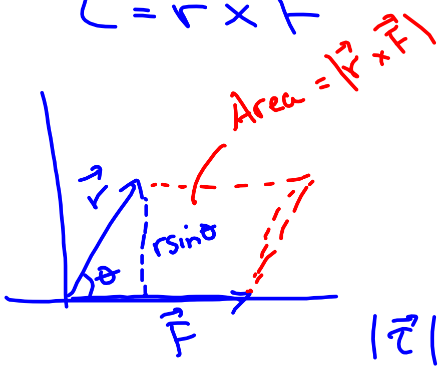
$$W = \vec{F} \cdot \vec{d} = \vec{d} \cdot \vec{F}$$

$$= F_{\parallel} d = F_x d$$

$$= F d \cos\theta$$

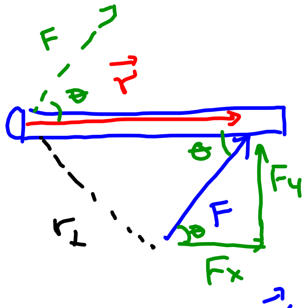
$$W = F_x d_x + F_y \cancel{d_y} + \cancel{F_z d_z}$$

recall




$$|\vec{\tau}| = |\vec{r}||\vec{F}|\sin\theta$$

$$\tau = r F_{\perp} = r_{\perp} F = r F \sin\theta$$

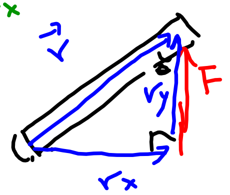



$$\tau = r F_y = r F \sin\theta$$

$r_{\perp}$ = moment arm

$$\tau = r_x F = r\sin\theta \cdot F$$

$$= r F \sin\theta$$




Units: $N \cdot m \neq J$ $\quad m \cdot N$

$$\vec{r} \times \vec{F} \neq \vec{F} \times \vec{r}$$

foot - pound

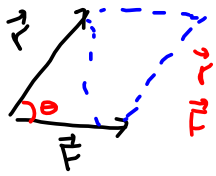
$$\vec{r} \times \vec{F} = -\vec{F} \times \vec{r}$$

10 N·m

10 mN mm

10 Nm

## Direction of Torque




Which way is the area facing?

$\vec{r}\times\vec{F}$ ⊗

$\vec{F}\times\vec{r}$ ⊙

$$\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ r_x & r_y & r_z\\ F_x & F_y & F_z\end{vmatrix} = (r_yF_z - r_zF_y)\,\hat{x} - (r_xF_z - r_zF_x)\,\hat{y} + (r_xF_y - r_yF_x)\,\hat{z}$$

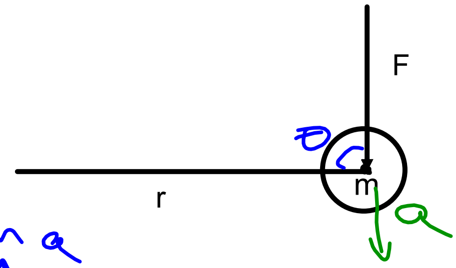
<u>RH rule</u> for $\vec{a}\times\vec{b}$

1) Fingers in direction of $\vec{a}$
2) Wrap your fingers in the direction of $\vec{b}$
3) Thumb ($\perp$ to both $\vec{a}, \vec{b}$) points in direction of $\vec{a}\times\vec{b}$

$\hat{x}\times\hat{y} = \hat{z}$ $\quad$ $\hat{x}\times\hat{x} =$ ☺ $\quad$ $\hat{y}\times\hat{x} = -\hat{z}$ $\quad$ $\hat{y}\times\hat{y} =$ ☺ $\quad$ $\hat{y}\times\hat{z} = \hat{x}$

$\hat{x}\times\hat{z} = -\hat{y}$ $\quad$ $\hat{z}\times\hat{x} = \hat{y}$ $\quad$ $\hat{z}\times\hat{y} = -\hat{x}$ $\quad$ $\hat{z}\times\hat{z} =$ ☺

## Torque on a Single Particle




$\theta = 90^\circ$

$\tau = rF\cancel{\sin\theta}^{\,1}$

$= rF$

$= rma$

$\Sigma\tau = \underbrace{mr^2}\,\underbrace{\alpha}$

moment of inertia

$I$

$\Sigma F = ma$

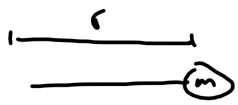
inertial mass (resistance to change)

$\alpha = \dfrac{a}{r}$

$a = r\alpha$

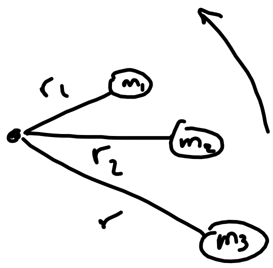
## Moment of Inertia

For a point particle



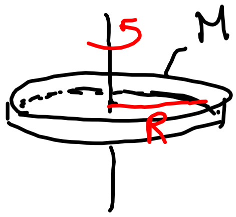

$$I = mr^2$$

For several point particles




$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2$$
$$= \sum m_i r_i^2$$

## Moment of Inertia - A Hoop



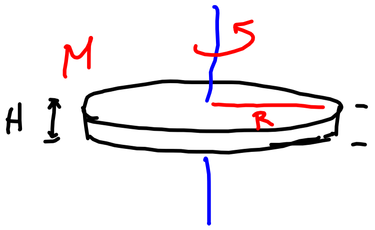

$I = \sum m_i r_i^2$ all at same radius R

$= \sum m_i R^2$

$= R^2 \underbrace{\sum m_i}$

$I = MR^2$

## Moment of Inertia - A Disc




$$I = \int_0^{2\pi}\int_0^{R}\int_0^{H} r^2 \underbrace{\overset{\text{density}}{\frac{M}{\pi R^2 H}}\, dh\cdot dr\cdot r\,d\theta}_{dm}$$

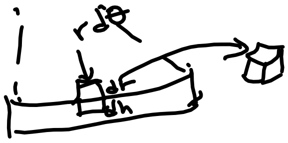
volume: $h \cdot l \cdot w$

$dV = dh\cdot dr\cdot r\,d\theta$

$$I = \sum m_i r_i^2$$

$$V = \pi R^2 H$$

$$\rho = \frac{M}{V} = \frac{M}{\pi R^2 H}$$



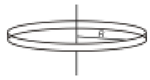
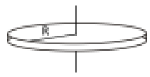
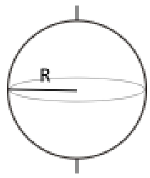
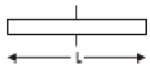
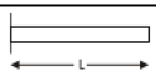

$$I = \frac{M}{\pi R^2 H}\int_0^{2\pi}\int_0^{R}\int_0^{H} r^3\, dh\, dr\, d\theta$$

$$= \frac{M\cancel{H}}{\pi R^2 \cancel{H}}\int_0^{2\pi}\int_0^{R} r^3\, dr\, d\theta$$

$$= \frac{M}{\pi \cancel{R^2}}\left(\frac{R^{\cancel{4}\,2}}{4}\right)\int_0^{2\pi} d\theta$$

$$= \frac{2\cancel{\pi} M R^2}{4\cancel{\pi}}$$

$$\underline{\underline{I = \frac{1}{2} M R^2}}$$

| Object | Axis of Rotation | | Moment of Inertia |
|---|---|---|---|
| Thin ring of radius $R$ and mass $M$ | Through center |  | $MR^2$ |
| Cylinder of radius $R$ and mass $M$ | Through center |  | $\frac{1}{2}MR^2$ |
| Sphere of radius $R$ and mass $M$ | Through center |  | $\frac{2}{5}MR^2$ |
| Long rod of length $L$ and mass $M$ | Through center |  | $\frac{1}{12}ML^2$ |
| | Around end |  | $\frac{1}{3}ML^2$ |