Projectile Motion and 2-D Dynamics

Vector Notation

Vectors vs. Scalars

In Physics 11, you learned the difference between vectors and scalars. A ***vector*** is a quantity that includes both direction and magnitude (size), whereas a ***scalar*** has no direction associated with it. Displacement, velocity, acceleration and momentum, among others are examples of vector quantities. Energy, distance and speed are some examples of scalars.

Last year you focussed primarily on vectors in one dimension. This year we will focus on 2- and 3-D vectors.

Bearings

One means of indicating direction that you used last year were bearings. If we consider a NSEW grid, 000° is the northerly direction, 090° easterly, 180° southerly and 270° westerly. The bearing is typically notated as a 3 digit (before the decimal place) angle. We can also give the direction of the vector as S30°E or E60°S.

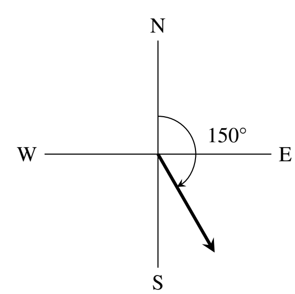

Figure 2.1: Vector with a bearing of 150°

Unit Vectors in 2D

As an alternative to giving a bearing, we can define a vector in terms of its components. On a Cartesian grid (x-y) we can define a vector as being so much in the x-direction and so much in the y-direction. To do this, we define *unit vectors*: a vector of length 1 in each direction. We use $\hat{\imath}$ or $\hat{x}$ as the unit vector in the x-direction and $\hat{\jmath}$ or $\hat{y}$ as the unit vector in the y-direction.

For example, we can consider a force vector $\vec{F} = 100$ N right and downward 60°. When considering Figure 2.2 we see that the vector is pointing both in the $+\hat{x}$ direction and in the -$\hat{y}$ direction. By knowing that the angle between the x-axis and the vector, we can calulate the components:

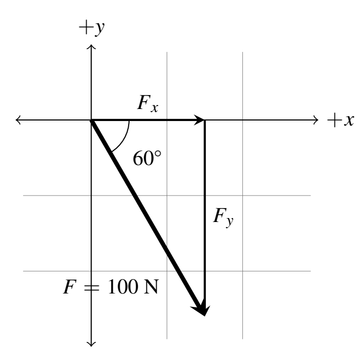

Figure 2.2: The vector above can be written in terms of its components as $\vec{F} = 50$ N$\hat{x}$ − 87 N$\hat{y}$.

$$F_x = F\cos 60^\circ = 50\text{ N};\;\; \mathrm{F_y} = \mathrm{F}\sin 60^\circ = 87\text{ N}$$

So the force vector can be written as

$$\vec{F} = 50\text{ N}\hat{\mathrm{x}} - 87\text{ N}\hat{\mathrm{y}}.$$

Note that the negative sign indicates the downward direction for y.

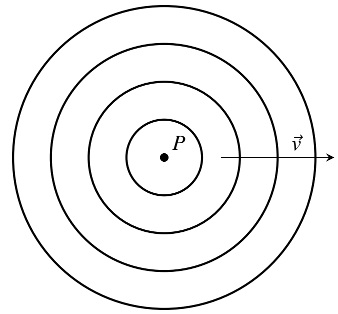

Figure 2.3: Waves moving outward from a point P at a velocity $\vec{v}$

IT IS SOMETIMES CONVENIENT to use coordinate systems that are different than the Cartesian plane, particularly for things which posess some symmetry. It 2D, for things that posess a circular symmetry, such as waves moving outward from a point on a surface. These waves form a circular pattern and move outward radially, as seen in Figure 2.3. The direction in which these waves are moving depends on *which point* on the wave we are talking about. If it is to the right of P it is moving in the $+\hat{x}$ but if it is above P it is moving in the $+\hat{y}$ direction. Clearly, this is not a convenient way to discuss the direction of the motion!

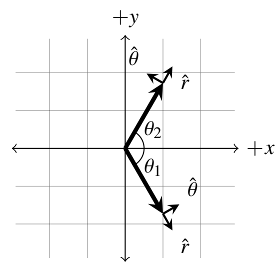

Figure 2.4: Unit vectors in polar coordinates.

We can describe this motion in terms of *polar coordinates*. In this case, the two "coordinates" are a radius, r, and an angle from the positive x-axis, θ. The unit vector $\hat{r}$ is in the radial outward direction. In specific cases, $\hat{r}$ is a function of (dependent upon) θ. The unit vector $\hat{\theta}$ is perpendicular to $\hat{r}$ and in the direction of increasing θ as shown in Figure 2.4. We can see that as θ changes from θ_1 to θ_2 the direction of the unit vectors change – however, $\hat{r}$ is *always* radially outward.

Referring back to the waves, since they are travelling radially outward, we can say that $\vec{v} = v\hat{r}$, where v is the speed.

Vectors in 3-D

WE CAN DESCRIBE VECTORS IN 3-D in a similar fashion to 2-D, but adding a third axis, the z-axis. In Cartesian coordinates, we can describe a vector as the sum of three components, for example

$$\vec{F} = (\hat{x} + 2\hat{y} + 3\hat{z})\ \text{N}$$

where the $\hat{z}$ refers to a unit vector in the $+z$-direction.

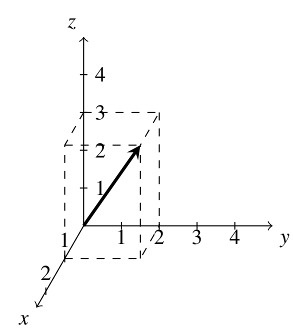

Figure 2.5: A vector in Cartesian 3-space: $\vec{F} = (\hat{x} + 2\hat{y} + 3\hat{z})$ N

JUST AS IT IS OFTEN CONVENIENT to use other coordinate systems in 2-D, it is useful to have alternate coordinate systems in 3-D. Two of these systems are *cylindrical coordinates* and *spherical coordinates*.

Cylindrical coordinate systems are useful for things that exhibit a cylindrical symmetry, such as a long straight wire carrying current. The current causes a magnetic field to be formed in circles around the wire. This is a topic of discussion for AP Physics.

Spherical coordinate systems are useful for things exhibiting a spherical symmetry. This would include things such as the Earth's gravitational field which is always (approximately) inwardly directed toward the center of the Earth (an approximate sphere). It is also useful for describing the electric field (a topic for later this semester) around a charged particle.

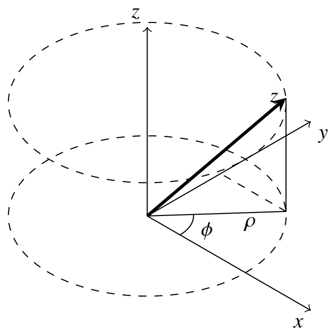

Figure 2.6: Vector in cylindrical coordinates.

Cylindrical coordinates are similar to polar coordinates in 2-D, but with a z-axis through the plane. In addition, we replace θ with ϕ (for consistency with spherical coordinates) and r with ρ, so a point in cylindrical coordinates is represented as (ρ, ϕ, z).

The unit vectors, as in cartesian coordinates, are in the direction of the increasing dimesion. So, $\hat{\rho}$ is in the direction of increasing ρ in the x-y plane, $\hat{\phi}$ is in the direction of increasing ϕ in the x-y plane and $\hat{z}$ in in the direction of the $+z$-axis.

Spherical coordinate systems are extremely useful, but we will only be using the radial component in this class. A point in spherical coordinates is defined by a radius, r, an angle θ from the $+z$ axis and an angle ϕ in the x-y plane from the $+x$ axis. In Figure 2.7 we see a vector from the origin to point P described by the three coordinates (r, ϕ, θ).

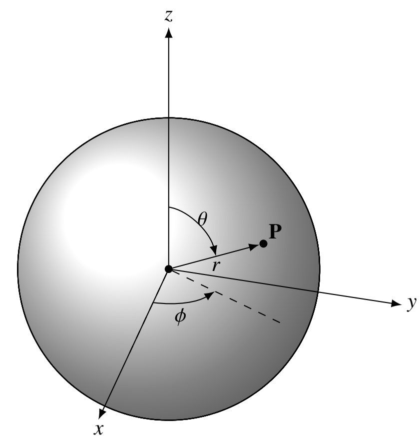

Figure 2.7: A vector in spherical coordinates

Vector Operations

IN PHYSICS 11, you learned a little bit about how vectors work. We will look at vector operations both from a graphical perspective as well as an algebraic one. Let's assume that we have a vector with arbitrary units $\vec{v} = 2\hat{x} + 3\hat{y}$. We can assume that the vector exists in the x-y plane in 3D or simply a 2D vector, but we will consider this example for ease of visualization.

One of the things we can do to a vector is stretch it – make it longer. This is done through ***scalar multiplication***. If we stretch $\vec{v}$ by a factor of 3, we have a vector $\vec{w} = 3\vec{v}$. This is illustrated in Figure 2.8. Algebraically, to multiply a vector by a scalar, we simply multiply each non-angular component by the scalar, so

$$3\vec{v} = 3(2\hat{x} + 3\hat{y}) = 6\hat{x} + 9\hat{y}.$$

In polar form, this vector has a length $r = \sqrt{2^2 + 3^2} = \sqrt{13}$ by Pythagoras, with an angle $\theta = \tan^{-1}\frac{3}{2} \approx 56.3°$, so $\vec{v} = \sqrt{13}\hat{r}$ where $\hat{r}$ is at an angle $\theta \approx 56.3°$. So

$$3\vec{v} = 3\sqrt{13}\hat{r} \text{ with } \theta \approx 56.3°.$$

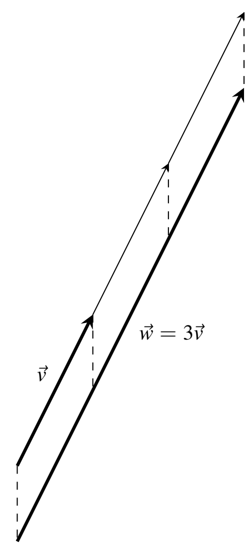

Figure 2.8: Scalar multiplication of a vector

When we add vectors, we can do this graphically using the ***tip-to-tail*** method of vector addition. We put the tip of one vector (the arrow head or point) next to the tail of another vector. We can see this illustrated in Figure 2.9 in which vector $\vec{v}$ is added to $\vec{w}$. Algebraically, it is also simple to add vectors in a Cartesian coordinate system - we just add the like components. That is $2\hat{x} + 3\hat{y}$ when added to $-\hat{x} + 2\hat{y}$ yields $(2 + (-1))\hat{x} + (3 + 2)\hat{y} = \hat{x} + 5\hat{y}$.

Subtraction of vectors is the addition of the negative, just as in scalars. The negative of a vector graphically is represented by the same line with the arrow pointed in the opposite direction. Algebraically, we simply subtract the like components in a Cartesian coordinate system. It should be noted that adding vectors in other coordinate systems can be more complicated than this and is an exercise for a later course.

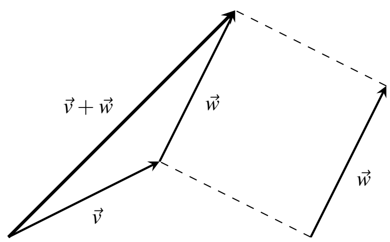

Figure 2.9: Tip-to-tail addition of $\vec{v}$ and $\vec{w}$.

THE MULTIPLICATION OF VECTORS does not exist in the sense that we think of scalar multiplication. There exist, however, two ***vector products***: the ***dot product*** and the ***cross product***. The dot product gives a scalar result whereas the cross product gives a vector result. Given a general vector $v_x\hat{x}+v_y\hat{y}+v_z\hat{z}$, if we dot it with itself, $\vec{v}\cdot\vec{v}$, we get the Pythagorean result, which is the length of the vector (or the norm of the vector) squared, i.e.,

$$\vec{v}\cdot\vec{v} = ||\vec{v}||^2 = v_x^2 + v_y^2 + v_z^2.$$

If we take the dot product with another general vector, $w_x\hat{x}+w_y\hat{y}+w_z\hat{z}$, we get

$$\vec{v}\cdot\vec{w} = v_xw_x + v_yw_y + v_zw_z.$$

This is due to the definition of the dot product which states

$$\vec{v}\cdot\vec{w} = ||\vec{v}||||\vec{w}||\cos\theta,$$

where θ is the angle between the two vectors. If the two unit vectors are orthogonal (at right angles), then the product is zero. If they are parallel, then the product is one. so $\hat{x}\cdot\hat{x}=1$, but $\hat{x}\cdot\hat{y}=\hat{x}\cdot\hat{z}=0$.

Work is in fact a dot product of the force and displacement vectors:

$$W = \vec{F}\cdot\vec{d} = Fd\cos\theta$$

as you learned last year.

The cross product is a vector product of two 3D vectors, which is defined by

$$\vec{v}\times\vec{w} = ||\vec{v}||||\vec{w}||\sin\theta\hat{n}$$

where θ is the angle between $\vec{v}$ and $\vec{w}$ and $\hat{n}$ is the unit vector perpendicular to $\vec{v}$ and $\vec{w}$ determined by the right hand rule. The right hand rule states that if you point your fingers in the direction of $\vec{v}$ and wrap them into $\vec{w}$, your thumb (when perpendicular to the two vectors) will point in the direction of $\hat{n}$.

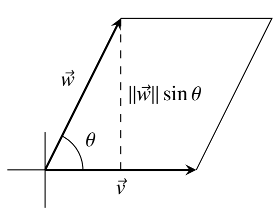

Figure 2.10: The cross product represented geometrically as the area of the parallelogram mapped by the vectors $\vec{v}$ and $\vec{w}$.

Geometrically, the cross product represents the area of the parallelogram mapped by the two vectors. In Figure 2.10 the area of the parallelogram is determined by the product of the base of the parallelogram and the height. In this case, the base is $||\vec{v}||$ and the height is $||\vec{w}||\sin\theta$. The product yields the size of the cross product. The direction (again determined by the right hand rule) is the direction the parallelogram is facing - in this case out of the page. (This is determined by wrapping $\vec{v}$ into $\vec{w}$ and pointing your thumb out of the page.)

From the definition of the cross product (or the geometric interpretation), we see that vectors that are parallel to each other result in a cross product of zero and vectors that are perpendicular to each other result in a vector with a length equal to the product of the lengths of the vectors (or the area of a rectangle). When considering the unit vectors, we see that $\hat{x} \times \hat{x} = \hat{y} \times \hat{y} = \hat{z} \times \hat{z} = 0$, while $\hat{x} \times \hat{y} = \hat{z}$, $\hat{y} \times \hat{z} = \hat{x}$, $\hat{z} \times \hat{x} = \hat{y}$ and $\hat{y} \times \hat{x} = -\hat{z}$, $\hat{z} \times \hat{y} = -\hat{x}$ and $\hat{x} \times \hat{z} = -\hat{y}$. This illustrates the anti-commutative property of the cross product (changing the order produces a negative result).

Projectile Motion

Sections 3-5 to 3-7.

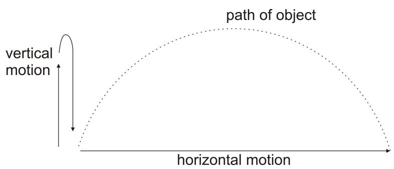

Figure 2.11: Projectile motion

A PROJECTILE INVOLVES AN OBJECT THAT HAS MOTION both in the up and/or down direction (vertical) as well as motion in the horizontal direction, such as a football pass (see Figure 2.11).

Recall from last year that for a boat going straight across a river that the sideways motion is unaffected by the current. That is the sideways motion is independent of the motion downstream. Likewise, with projectiles, the motion can be divided into two independent components: up/down (affected by gravity) and sideways (not affected by gravity).

The ball is thrown upward, and comes down because of the force of gravity. The ball also moves horizontally, and has no external forces affecting it in this direction after leaving the thrower's hand (ignoring air resistance). We can analyze the motion in each direction independently - as if it were a ball going straight up and down, and a ball going straight sideways - and then combine the information as vectors. To start, we will look at a simple example.

Air to Ground Projectile Motion (Horizontal Projectiles)

CONSIDER A PLANE CARRYING A BOMB at an altitude of 6000 m. The plane is flying horizontally at a constant velocity (say 200 m/s East). If the plane then releases the bomb and continues horizontally, at the same velocity, where does the bomb land (ignoring air resistance)? The answer is that it lands just underneath the plane!

If this doesn't make sense, consider a book sliding along the floor. It is only friction that causes it to slow down. If the book is sailing through the air in the absence of gravity, it is only air resistance (friction) that would make it slow down. So, in the case of our bomb, it is dropped with an initial horizontal speed of 200 m/s. There is no force in the horizontal direction causing it to either speed up or slow down in that direction (because we ignore air resistance). So the bomb will maintain the same horizontal velocity that it started with. If the plane continues at the same velocity, the bomb will always be directly under the plane.

To solve projectiles, I recommend that you divide your page in two columns: one column for the horizontal ($x-$) direction, and one column for the vertical ($y-$) direction, as shown below. In the $x-$direction, we know that the acceleration is 0 (v_x is constant), and in the $y-$direction, we know that the acceleration is g (the acceleration of gravity) downward.

We can determine how long it takes the bomb to fall simply by finding how long it takes any object to fall 6000 m (if $v_i = 0$). This time is the total time that the bomb is in the air, so it is the same in both the $x-$ and $y-$directions. We can also find the final velocity in the $y-$direction. Using the time and the horizontal velocity, we can determine how far the bomb travels horizontally.

x−direction	**y**−direction
$v_i = 200\ \frac{\text{m}}{\text{s}}$ East	$v_i = 0$
$a = 0$	$a = -g = 9.807\ \frac{\text{m}}{\text{s}}$
$t = ?$	$d = 6000$ m
	$d = v_i t + \frac{1}{2}at^2$, but $v_i = 0$
	$-6000 = \frac{1}{2} \times -9.807\ \frac{\text{m}}{\text{s}}$
	$t^2 = 1224\text{s}^2$
	$t = 34.99\text{s}$
$d = vt = \left(200\ \frac{\text{m}}{\text{s}}\right)(34.99s)$	
$d = 7.00 \times 10^3$ m East	So $v_f = v_i + at$
	$= 0 + \left(-9.807\ \frac{\text{m}}{\text{s}}\right)(34.99\text{s})$
**Note: the distance the projectile travels in the x−direction is called the *range*.	$= -343\ \frac{\text{m}}{\text{s}}$

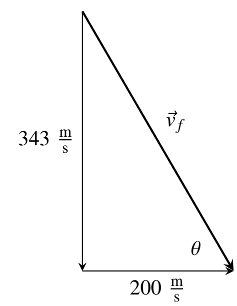

Figure 2.12: Final velocity by vector addition. The hypotenuse is calculated using Pythagoras and the angle is calculated using the arctangent.

If we wish to determine the velocity of the bomb as it hits the ground, it becomes a matter of vector addition: 200 $\frac{\text{m}}{\text{s}}$ horizontally added to a downward velocity of 343 $\frac{\text{m}}{\text{s}}$. The resultant vector is 397 $\frac{\text{m}}{\text{s}}$ at 59.8° to the ground (See Figure 2.12).

Ground to Ground Projectiles (Symmetrical Projectiles)

GROUND TO GROUND PROJECTILES are solved in the same manner as air to ground, except that the initial velocity in the y-direction is not zero (it wouldn't go very far if this were the case). This does provide a slightly greater difficulty in finding the time in the air than with air to ground projectiles. The procedure for finding this time is outlined below.

1. Since the projectile lands at the same height as it was launched, you can determine the time, t_{up}, it takes the projectile to reach its maximum height (knowing $v_{fy} = 0$ and given v_{iy}). Then multiply this time by two to obtain the total time, t in the air. We will prove this in class.
2. You can find the range using the total time, t and the horizontal velocity, v_x.
3. You can find the maximum height using half the total time.
4. With the initial velocity, time and maximum height, we can determine any values we like.

The symmetry in this case should not be ignored. Because the height up and down are equal, the times up and down are equal, and the projectile hits the ground with the same speed as it was launched and at the same angle.

Asymmetrical Projectiles and Energy

Sections 3-5 to 3-7.

Asymmetrical Projectiles

WE HAVE LOOKED AT THE TWO SIMPLE CASES of projectile motion, so it is time to look at a general projectile - one that lands at a different height from which it was launched with $v_{iy} \neq 0$. In this case, we can follow a similar procedure to symmetrical projectiles, except instead of doubling the time, we must calculate the time down, t_{down}. To get this, we must determine the maximum height, h_{max} and then the displacement the object falls. The remaining procedure is the same.

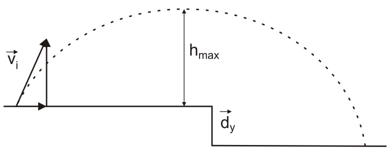

Figure 2.13: Asymmetrical projectile with a negative vertical displacement

We can also always calculate total time using the quadratic equation in the $y-$direction.

Since $d = v_i t + \frac{1}{2}at^2$

$d_y = v_{iy}t - \frac{1}{2}gt^2$

So $0 = \frac{1}{2}gt^2 - v_{iy}t + d_y$ which is the form of a quadratic

Giving $t = \dfrac{v_i \pm \sqrt{v_i{}^2 - 2gd_y}}{g}$ using the quadratic solution

It should be noted that this provides two solutions for time. Because the motion is parabolic in nature, there are two times when an object is at a specified height (other than h_{max}. You must determine which solution is valid in the given situation (if d_y is negative then one solution for time will be negative, so there's no problem).

Solving Projectiles Using Energy

SINCE WE ARE INVESTIGATING THE CASE in which air resistance is ignored, mechanical energy is conserved at each point in a projectile, that is

$$E_T = KE_1 + PE_1 = KE_2 + PE_2$$

where E_T is the total energy,KE_1 and KE_2 are the kinetic energies at points 1 and 2 respectively and PE_1 and PE_2 are the potential energies at those points.

If we know the initial velocity (speed and direction) of the projectile, we can determine v_x which is constant throughout the projectile. Using the conservation of energy, we can find the *speed* at any point, or the height at which a certain speed is reached (e.g. at $h_{max}, v = v_x$). To find the direction at a particular point, we know that $\cos\theta = \dfrac{v_x}{v}$.

The Power of Free Bodies

Section 4-7

When there is more than one object involved in a system, there are times it is convenient to look at the system as a whole, and there are other times it is convenient (and more powerful) to look at the forces on an individual object first. We shall look at the former first.

Consider two masses (mass m and M respectively), connected by a massless rope over a frictionless pulley (see 2.14). Obviously, the more massive object will "win", but what will be the acceleration?

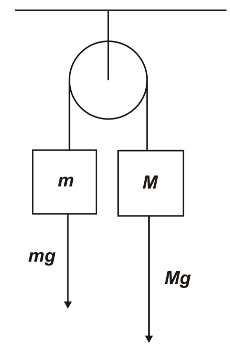

Figure 2.14: Atwood machine

In this case (we'll assume for convenience in discussion that $M > m$), mass M goes downward and pulls mass m upward. The total forces affecting the system are Mg pulling one way and mg pulling the other. The net force, $\Sigma F = Mg - mg$, pulls to the right hand side (or turns the pulley clockwise). The total mass being accelerated is $M + m$. Therefore the acceleration of this system is

$$a = \frac{\Sigma F}{m+M} = \frac{M-m}{m+M} g \text{ clockwise.}$$

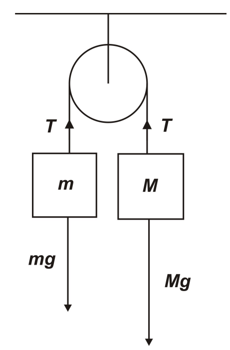

Figure 2.15: Atwood machine

But let's change the question slightly. In this case, let's say we have mass M connected to the end of a rope that will only hold a weight, T (we use T for tension), before it breaks (see Figure 2.15). What mass, m, on the other end of the rope will provide the smallest acceleration downward for mass M without the rope breaking?

In this case, we must look at each mass individually, and the rope as well! We know $m < M$ if mass M is supposed to accelerate downward. We also know that the most the rope can hold is T without breaking. In order for the large mass to be accelerating as little as possible, the rope must be doing as much as possible. So, the net force on mass M must be downward. This means the acceleration of M is

$$a = \frac{Mg - T}{M}.$$

Since a rope connects both masses and the rope is pulling, we know the two masses must be accelerating the same! We also know (from Newton's 3rd law) that the rope is pulling the same all the way around (it can not pull harder in one direction than another), so it is pulling upward on m with a force T. From our diagram we can see that the net force on block m is $\Sigma F = T - mg$, and the acceleration must be $a = \frac{T-mg}{m}$. But the two accelerations must be equal, giving $\frac{Mg-T}{M} = \frac{T-mg}{m}$. With a little work, we can solve for m: $m = \frac{MT}{2gM-T}$.

While dealing with free body diagrams (each body individually) is mathematically more intricate, it is the most reasonable way to deal with more complicated problems.

We will look at an example of a moveable pulley as well.

Forces in 2-Dimensions: Inclined Planes

Section 4-8

We will now consider what happens when an object is resting on a slope, as opposed to a level surface. If we investigate a frictionless slope making an angle of θ with the ground, we know the object will slide downhill. Why? The answer is gravity - or at least a component of it.

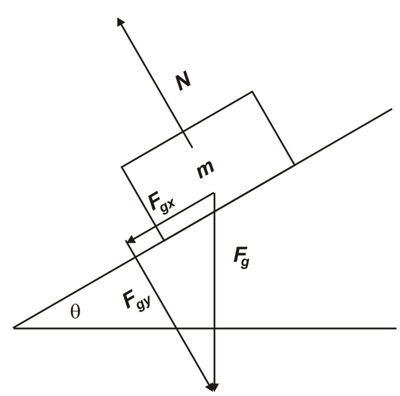

Figure 2.16: Free body diagram of a mass on an inclined plane

Gravity, as always, acts straight down. But we can break the force up into a component pulling the object down the hill (F_{gx}) and a component pushing it into the surface (F_{gy}). What we have effectively done is tilted our $x-$ and $y-$axes.

In the diagram above, you can see the result: the normal force is equal in size to the force pushing the object into the surface (F_{gy}), while F_{gx} is accelerating the object down the hill.

Beyond this, we can add friction and forces pulling up or downhill, or even at an angle to the hill, but the principles remain the same.

Examples: Pulling uphill, downhill, another way to calculate coefficient of friction.

Other Applications

This will be a day of examples - seeing how Newton's laws apply to more complex situations Examples looking at energy conservation and inclines (with/without friction). Multiple bodies - horizontal and stacked bodies with friction, inclines with pulleys, etc. - including some SIN questions.