Circular Motion and Universal Gravitation

Uniform Circular Motion

Sections 5-1 to 5-2

An object travelling in a circle with a constant speed, v, is said to be experiencing uniform circular motion. While the *speed* of the object is constant, the *direction* and thus the *velocity* is constantly changing, so the object is accelerating. This acceleration is called the centripetal acceleration, as is determined by

$$\vec{a}_c = \frac{d\vec{v}}{dt} \tag{3.1}$$

where $\frac{d\vec{v}}{dt}$ represents the derivative or change in velocity with respect to time. It is essentially the slope of a velocity-time graph at any point in time.

We can derive this geometrically, by looking at looking at the velocity at two points in time, v and v' as indicated in Figure 3.1. Since the speed is constant, $v = v'$. Comparing the r-r-Δd triangle with the v-v'-Δv triangle, they are similar, so

$$\begin{aligned} a_c = \frac{\Delta v}{\Delta t} &= \frac{\frac{v \Delta d}{r}}{\Delta t} \\ &= \frac{v}{r}\frac{\Delta d}{\Delta t} \\ &= \frac{v^2}{r} \end{aligned} \tag{3.2}$$

since $\frac{\Delta d}{\Delta t} = v$. Note that the direction is the same as that of Δv, which, as $t \to 0$ is directed inward, toward the center of the circle.

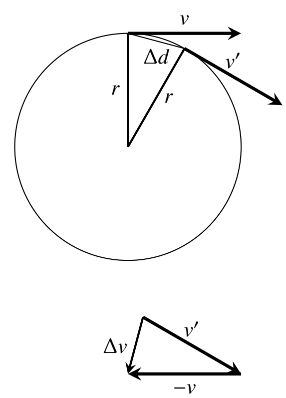

Figure 3.1: Velocity and displacement triangles in centripetal acceleration

An object travelling at a constant speed in a circle travels the circumference ($C = 2\pi r$) in the same amount of time for each circle. This time, T, is called the period. An object that repeats its motion over and over again is said to experience periodic motion. The speed of the object travelling in the circle, then, can be determined by

$$v = \frac{d}{t}$$

So

$$v = \frac{2\pi r}{T}.$$

We can substitute this value of v into equation 3.2 and get

$$\begin{aligned} a_c &= \frac{\left(\frac{2\pi r}{T}\right)^2}{r} \\ &= \frac{\frac{4\pi^2 r^2}{T^2}}{r} \\ &= \frac{4\pi^2 r}{T^2} \end{aligned} \tag{3.3}$$

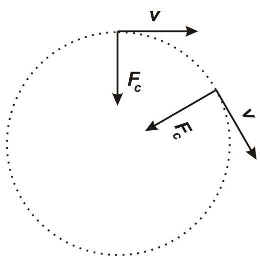

Figure 3.2: Circular motion indicating velocity and centripetal force

NEWTON'S SECOND LAW TELLS US THAT IF THERE IS AN ACCELERATION, it is caused by a force. This ***centripetal force*** (an inward pulling force) is given by

$$F_c = \frac{mv^2}{r} \tag{3.4}$$

or

$$F_c = \frac{m4\pi^2 r}{T^2}. \tag{3.5}$$

During circular motion, at any given time, the velocity of the object is perpendicular to the radius, and the centripetal force acts inward along the radius (see Figure 3.2). This is easily observed when swinging a weight attached to the end of a string in a horizontal circle. If you were to cut the string (or if the string were to break), the weight would fly off in its direction of motion at the time (perpendicular to the string). What keeps the weight from travelling in a straight line? The string does, by constantly providing an inward force. Note that the centripetal force is simply the net force ($\Sigma\vec{F}$) acting toward the center of a circle.

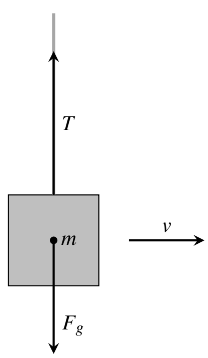

Figure 3.3: At the bottom, the tension in the rope (handle or arm) holds the pail up, while gravity pulls it down. In order to make it swing upwards in a circle with speed v, T must be larger than F_g by $\sum F = T - F_g = \frac{mv^2}{r}$

Non-Uniform Circular Motion

THE CASE OF AN OBJECT WHICH IS SPEEDING UP (or slowing down) while it travels in a circle is called ***non-uniform circular motion***. In this case, we have the acceleration caused by the circular motion (equations 3.2 and 3.3), but we also have ***tangential acceleration*** (a_T). Tangential acceleration is calculated using your one-dimensional equations of motion. To determine the total acceleration, we must add these two accelerations as vectors.

Vertical Circles and Banked Turns

Sections 5-2 to 5-3

Vertical Circles

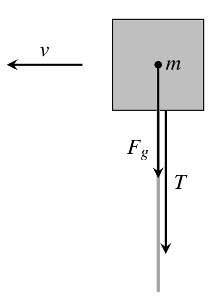

Figure 3.4: At the top, however, the string is (obviously) pulling downward, as is gravity. In this case, T and F_g add to give the net force, so $\sum F = T + F_g = \frac{mv^2}{r}$. As long as v is sufficiently great for $T > 0$, the water will stay in the pail. Otherwise you get wet.

AS OPPOSED TO HORIZONTAL CIRCLES, with vertical circles, we must contend with the force of gravity. Consider the simple case of swinging a bucket full of water upside down (which amazed you all as children when the water *didn't* fall out). Look at the free body diagrams at the bottom (Figure 3.3) and the top (Figure 3.4) of the circle.

Notice that this changes if we are dealing with a rigid structure (like a Ferris wheel) instead of a rope. The rigid structure can push upward, and so the wheel doesn't have to be travelling quickly in order to maintain balance at the top. (We will investigate this example in class).

Banked Curves

COMBINING HORIZONTAL CIRCLES WITH INCLINES is slightly nasty, but not impossible. Of course civil engineers deal with this all the time in road design. Very seldom do you travel on a highway around a turn without it being banked somewhat. This is simply because the increased normal force toward the center of the circle decreases the amount of friction required to make the turn.

Assume a mass, m is travelling around a banked turn (radius r) with speed v (into the page) and that this speed is fast enough that he would tend to slide up the hill, so friction acts down the hill (see Figure 3.4).

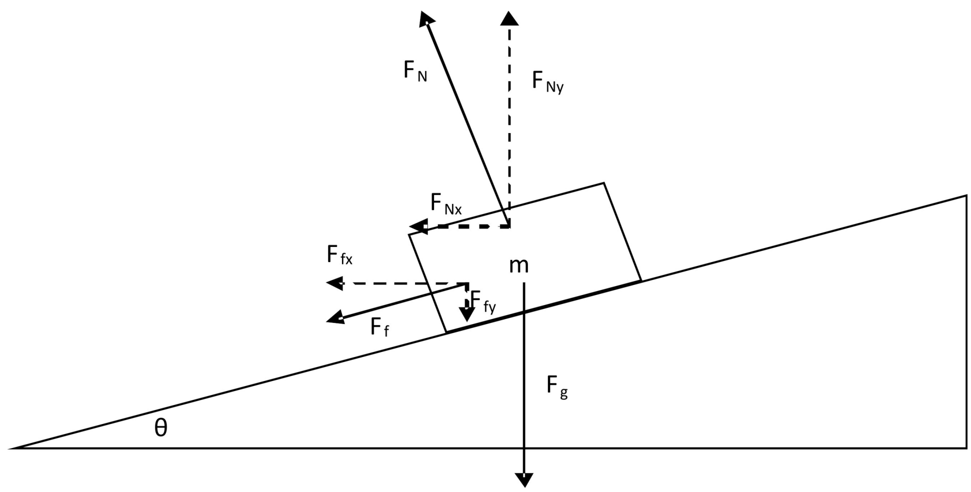

Figure 3.5: Free body diagram of a mass going around a banked turn

Then we can see that $\sum F_y = 0$ (since the car moves neither up nor down the hill) and $\sum F_x = \frac{mv^2}{r}$ to maintain a constant radius. Using these equations we get

$$F_{Ny} = F_{fy} + F_g \qquad \text{and} \qquad \frac{mv^2}{r} = F_{Nx} + F_{fx}$$

The first equation allows us to calculate F_N: $F_N = \frac{mg}{\cos\theta - \mu\sin\theta}$. Substituting this into the second equations gives us the maximum velocity without slipping uphill:

$$\frac{mv^2}{r} = mg\left(\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}\right).$$

Not surprisingly, the masses cancel. Solving the equation for v gives

$$v = \sqrt{gr\left(\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}\right)}.$$

Notice that in the trivial case of $\theta = 0$, $v = \sqrt{\mu gr}$ as we would expect on a flat surface.

Kepler's Three Laws of Planetary Motion

Section 5-9

Introduction

BEFORE THE INVENTION OF THE TELESCOPE, there were seven heavenly bodies moving amongst the stars that were known to man: the sun, the moon, Mercury, Venus, Mars, Jupiter and Saturn, which could all be seen with the naked eye.

Early planetary systems fell into two basic categories: the egocentric view, that the earth is the center of the solar system (or universe) which was modeled by Ptolemy among others; and the heliocentric view, that the sun is the center of the solar system, modeled by Copernicus (b. 1473). Of course, we hold to the latter, but the reasons for this are not obvious. It should be noted that there were ancient Greeks, such as Heracleides and Aristarchus who thought that the earth rotated on its axis and that it moved around the sun.

The seriousness of Copernicus' model must be understood by its place in history. In order to develop his model, he had to discard the entire picture of the universe as it had been developed since Aristotle. If, in fact, the earth moves, it destroys many of the ideas of motion which were understood. For example, imagine a bird on top of a tree that sees a worm at the base. If it takes one second for the bird to reach the ground, and the earth is moving (which it would have to do at a speed of about 30 km/s orbiting the sun), then the bird would have to travel 30 km to catch the worm! But there were other arguments as well, such as, "If the earth is moving, what is pushing it?"

A Danish astronomer, Tycho Brahe, (b. 1546) could not accept the Copernican model, and set about developing a highly accurate map of the stars. With these measurements, he purposed to find a more accurate description of the orbits of the heavenly bodies. However, he was an old sot and drank himself to death (in 1601, in a rather unusual and horrible manner, according to legend) before he could accomplish this.

Kepler's Laws of Planetary Motion

BRAHE'S STUDENT, JOHANNES KEPLER (b. 1571), was very mathematical, and unlike Brahe, believed in the Copernican model. Comparing a mathematical model of circular orbits to Brahe's measurement, he found that they were close, but not close enough. He then discovered that the planets moved around the sun, not in circular orbits, but in elliptical ones.

Kepler, with this discovery, wrote down what we now refer to as Kepler's laws of planetary motion.

Kepler's first law: The path of each planet about the sun is an ellipse with the sun at one focus.

"***Kepler's second law:*** Each planet moves so that an imaginary line drawn from the sun to the planet sweeps out equal areas in equal periods of time."[1] As seen in figure 3.6, the two areas are approximately equal, and occur in the same period of time. This means that the orbiting body travels faster when it is closer to the body being orbited.

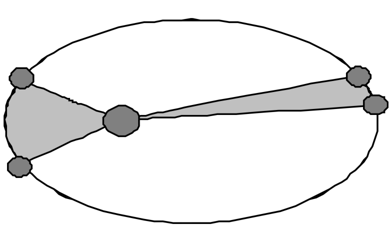

Figure 3.6: Kepler's second law

[1] Giancoli, p. 115

"***Kepler's third law:*** The ratio of the squares of the periods of any two planets revolving about the sun is equal to the ratio of the cubes of their mean distances from the sun. That is, if T_1 and T_2 represent the periods, and r_1 and r_2 represent their average distances from the sun, then" [2]

[2] ibid.

$$\left(\frac{T_1}{T_2}\right)^2 = \left(\frac{r_1}{r_2}\right)^3, \tag{3.6}$$

or for any objects revolving around the same body

$$K = \frac{r^3}{T^2} \tag{3.7}$$

where K is Kepler's constant for a body. This means that for any planet orbiting the sun, K is constant, but for a moon of Jupiter, the constant is different.

Newton's Law of Universal Gravitation

Sections 5-6 to 5-8

ACCORDING TO LEGEND, Newton was sitting under an apple tree in his orchard and a falling apple caused him to ponder the force which pulled the apple to the ground. Could the force of attraction between the apple and the ground be the same as the force which held the planets in orbit around the sun? The rest, as they say, is history.

Newton, using Kepler's work as a starting point, showed mathematically that if Kepler's 3rd law were true, then the size of the gravitational force must be inversely proportional to the distance between the two planets, r, squared, or $F \propto \frac{1}{r^2}$. Reasoning that the force required to accelerate a massive object in a circle must be greater than that required to accelerate a less massive object, he also concluded that the size of the gravitation force must proportional to the mass (since $F = ma$) of the planet (m_1). But similarly, it must also be proportional to the mass of the sun or planet being orbited (m_2). So, Newton proposed that the gravitational force between two objects is equal to

$$F = \frac{Gm_1m_2}{r^2}, \tag{3.8}$$

where G is a universal gravitational constant. This is known as ***Newton's Universal Law of Gravitation***.

At the time, this was useful only for comparison purposes. The mass of the earth and the sun were unknown, as was G. It took about 100 years before the gravitational constant was calculated. It was done in 1798 by Henry Cavendish. He took a rod, having a lead ball on each end, suspended from a thin wire and by measuring the amount the rod twisted when the lead balls came close to other balls, he was able to determine the gravitational force of attraction. Since Cavendish knew the size of the force, the distance and the two masses, he could easily solve for G.

The modern value for G, when the masses are measured in kilograms, the distance in meters, and the force in Newtons is

$$G = 6.67 \times 10^{-11}\,\frac{\mathrm{Nm}^2}{\mathrm{kg}^2}.$$

By assuming circular orbits for the planets (which they nearly are), we can equate the gravitational force with the centripetal force, since it is the gravitational force which causes the planets to go in their circular orbits. This gives us

$$G\frac{m_1m_2}{r^2} = m_1\frac{4\pi^2 r}{T^2} \tag{3.9}$$

where r is the radius of orbit, T is the period of orbit, m_1 is the mass of the orbiting planet and m_2 is the mass of the body being orbited. By cancelling m1 and rearranging for T^2, we get

$$T^2 = \left(\frac{4\pi^2}{Gm_2}\right) r^3$$

or

$$\frac{r^3}{T^2} = \frac{Gm_2}{4\pi^2} = K. \tag{3.10}$$

So the Kepler constant depends upon the mass of the object being orbited. And knowing G enables us to determine the mass of the sun (or any orbited planet).

Weighing Earth

KNOWING THE GRAVITATIONAL CONSTANT, the acceleration of gravity on the surface of the earth and the radius of the earth, it is possible to determine the mass of the earth. We can equate the force of gravity on earth with the universal equation and we obtain

$$mg = \frac{GmM_E{}^2}{r_E}$$

where m is the mass of an object on the planet, M_E is the mass of the earth and r_E is the radius of the earth (we take the distance between the centers rather than the distance between the surfaces). Solving for the mass of the earth we get

$$M_E = \frac{g r_E{}^2}{G}. \tag{3.11}$$

When we substitute values and calculate, we determine that the mass of Earth is 5.98×10^{24}kg

By using values of moons for different planets, we can equate centripetal acceleration to acceleration of gravity and use the radius of orbit of the moon around the planet, we can use equation 3.11 to determine the mass of *any* planet with a satellite by

$$M_p = \frac{a_g r^2}{G}. \tag{3.12}$$

Gravitational Potential Energy and Escape Velocity

LAST YEAR WE TREATED POTENTIAL ENERGY DUE TO GRAVITY as $PE = mgh$ (which comes from the work required to lift an object to height h). This, however, assumes a constant gravitational field, g.

Potential energy due to gravity becomes much more complicated when the gravitational field is not uniform. This happens, of course, as we get further and further from a planet. We know that the acceleration of gravity is equal to

$$a_g = \frac{GM}{r^2}$$

where M is the mass of the planet from Newton's law of gravitation. If r changes significantly (about a hundred km or more for earth), then we cannot consider the gravitational field to be constant, so we cannot use equation $PE = mgh$. Instead, this becomes a calculus problem.

The first problem is where to choose for our zero point. Surprisingly, we know at an infinite distance away from the planet, the acceleration is zero, and so $r = \infty$ is a good choice for a zero potential energy (it won't fall anywhere from an infinite distance away). From there, we determine the work required to get an object that far away:

$$W = F_{\|}d \tag{3.13}$$

but $F_{\|}$ is constantly changing. The only way to solve this is with integration(which is basically adding the areas under the graph!):

$$W = \int_{r_0}^{\infty} \frac{GMm}{r^2}\,dr \tag{3.14}$$

We will discuss this in class as to what this actually means in layman's terms, but it is basically the area under the graph from some radius r_0 to $r = \infty$ as illustrated in Figure 3.7. If we determine the work required to lift an object this far, to overcome gravity, we know the potential energy is equal to this. Solving the above equation gives us the equation for potential energy relative to infinity:

$$PE(r) = -\frac{GMm}{r} \tag{3.15}$$

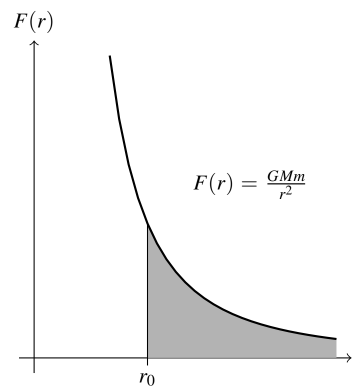

Figure 3.7: Area under $\frac{GMm}{r^2}$ from r_0 to $r = \infty$

Example

HOW MUCH ENERGY DOES IT TAKE to lift a 1000 kg satellite from earth's surface to an altitude of 500 km?

The amount of energy required is simply the change in potential energy of the satellite. So,

$\Delta PE = PE_f - PE_i$
$M = 5.98 \times 10^{24}$ kg
$m = 1000$ kg

$r_f = 6.38 \times 10^6 + 5.00 \times 10^5$ kg $= 6.88 \times 10^6$ m
$r_i = 6.38 \times 10^6$ m

Giving

$$PE_f = -\frac{GMm}{r_f} = -\frac{\left(6.67 \times 10^{-11}\ \frac{\text{Nm}^2}{\text{kg}^2}\right)\left(5.98 \times 10^{24}\ \text{kg}\right)(1000\ \text{kg})}{6.88 \times 10^6\text{m}} = -5.797 \times 10^{10}\ \text{J}$$

$$PE_i = -\frac{GMm}{r_i} = -\frac{\left(6.67 \times 10^{-11}\ \frac{\text{Nm}^2}{\text{kg}^2}\right)\left(5.98 \times 10^{24}\ \text{kg}\right)(1000\ \text{kg})}{6.38 \times 10^6\ \text{m}} = -6.252 \times 10^{10}\ \text{J}$$

So the energy required is
$\Delta PE = -5.797 \times 10^{10}\ \text{J} - -6.252 \times 10^{10}\ \text{J} = 4.55 \times 10^9\ \text{J}$

Escape Velocity

ESCAPE VELOCITY is the velocity required to "escape" the field of attraction of some object. This could be an electric field, a magnetic field, or in our case a gravitational field. We are essentially trying to calculate the velocity which would, in absence of other forces, take us to $r = \infty$. At any point relative to the object, we have a potential energy $PE = -\frac{GMm}{r}$. To reach $r = \infty$, we require a total energy of 0. Therefore the KE required to achieve this is $KE = \frac{GMm}{r} = \frac{1}{2}mv^2$. The velocity determined by this is known as the escape velocity. In our case, $v_{esc} = \sqrt{\frac{2GM}{r}}$.