Rotational Motion and Torque

Introduction to Angular Quantities

Sections 8-1 to 8-2

Introduction

ROTATIONAL MOTION deals with spinning objects, or objects rotating around some point. Rotational motion is analogous to linear motion, and torque is analogous to force. To deal with rotational motion, however, we must define new quantities. The equations dealing with these quantities, though, are identical in format to those for linear motion.

Angular Quantities

THE BASE QUANTITY IN ROTATIONAL MOTION is the angle. If an object is spinning, it is rotated through an angle, and a complete circle is, of course, 360°. The unit degree, however, is a very inconvenient unit for dynamics, and so we must define a more appropriate one: the radian.

A ***radian*** is defined as the angle whose subtended arc is equal to the radius. That is $\frac{\ell}{r} = 1$ (see Figure 4.1). The angular displacement, θ, is defined as

$$\theta = \frac{\ell}{r}. \tag{4.1}$$

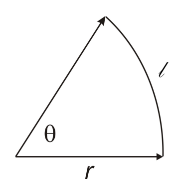


Figure 4.1: Arc length, radius and the angle in radians

The angular displacement is analogous to linear distance. There are 2π radians in 360°.

ANGULAR VELOCITY, ω (the Greek lowercase omega), is defined in analogous terms to linear velocity, except instead of using displacement, we use angular displacement. That is

$$\omega = \frac{d\theta}{dt} \tag{4.2}$$

This equation can be used for average angular velocity (if the angular speed is changing) or the instantaneous angular velocity for constant ω, or small t. Much like we can plot a d-t graph and find the slope to get velocity, we can plot a θ-t graph and find the slope to get ω.

ANGULAR ACCELERATION, α (the Greek letter alpha), is defined like linear acceleration:

$$\alpha = \frac{d\omega}{dt} \approx \frac{\omega_f - \omega_i}{t} \tag{4.3}$$

where $\Delta\omega$ is the change in angular velocity, ω_f is the final angular velocity and ω_i is the initial angular velocity.

Because there is a direct transformation between linear and rotational motion, you can use all of your linear motion equations with rotational quantities. For example:

$$\theta = \omega_i t + \frac{1}{2}\alpha t^2 \tag{4.4}$$

and

$$\omega_f{}^2 = \omega_i{}^2 + 2\alpha\theta. \tag{4.5}$$

WE CAN RELATE ANGULAR VELOCITY with linear velocity. By combining equations 4.1 and 4.2 we can show that any object spinning at a radius r with an angular velocity ω that

$$\omega = \frac{\ell}{rt}$$

but $\frac{\ell}{t}$ is just distance over time, which is the linear speed, v. This gives us

$$v = r\omega \tag{4.6}$$

with the direction of the velocity being tangent to the radius. Similarly, it can be shown that the tangential acceleration can be related to the angular acceleration by

$$a = r\alpha \tag{4.7}$$

(As a good exercise, you should convince yourself of this. Hint: use equations 4.3 and 4.6.)

Rotational Dynamics and Torque

Sections 8-3 to 8-5

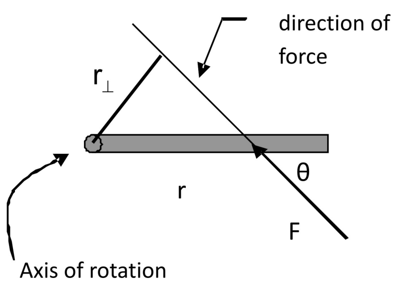


Figure 4.2: The moment arm of a force acting on a rod

JUST AS FORCE IS RESPONSIBLE for a linear acceleration, so force is also responsible for a rotational acceleration. However, not only is the angular acceleration proportional to the force, it depends on the ***moment arm*** and direction of that force.

The moment arm, $r_\perp$, is the perpendicular distance from the line on which the force acts to the axis of rotation (see Figure 4.2). We can see that if the force makes an angle θ with the rod, then

$$r_\perp = r\sin\theta \tag{4.8}$$

where r is the distance from the axis of rotation to the point where the force is applied.

THE PRODUCT OF THE MOMENT ARM AND THE FORCE is called ***torque***, (τ, the Greek letter tau) and it is the torque which causes the angular acceleration associated with the force. This makes sense if you have tried to close a door by applying a force at different points. The closer you are to the hinge, the harder you must press to make the door close the same amount. (This is exactly the same as a lever, and is why wrenches have long handles). Torque therefore, is calculated by

$$\tau = r_\perp F$$

or

$$\tau = rF\sin\theta. \tag{4.9}$$

This is in fact a vector cross product (as we saw in chapter 2) represented in the form $\vec{\tau} = \vec{r} \times \vec{F}$, and the direction of the torque is perpendicular to both $\vec{r}$ and $\vec{F}$.

In Figure 4.2, you can also see that you obtain the maximum moment arm (and therefore the maximum torque) when θ is 90°, or when you push perpendicular to the object. This is reflected in equation 4.9, since τ is a maximum when $\sin\theta = 1$.

Much like there is a direct relationship between force and acceleration, there is a direct relationship between torque and angular acceleration. Combining $F = ma$ with equations 4.7 and 4.9, we find that

$$\tau = mr^2\alpha \tag{4.10}$$

for a single particle, where r is the radius of the circle (i.e. the moment arm).

Moment of Inertia

THE QUANTITY mr^2 from equation 4.10 represents a quantity, which is the rotational inertia of the particle and is called the ***moment of inertia***, I. The quantity that represents linear inertia is mass, and so moment of inertia is the rotational equivalent of mass. Knowing $I = mr^2$ for a single particle, we can substitute this in equation 4.10 and get

$$\tau = I\alpha \tag{4.11}$$

The moment of inertia, however becomes much more complicated for rotating rigid objects (such as a wheel, which is a disc that rotates about its center). In these cases, the mass is distributed at different radii, and so the radius is not constant. We must sum up the mr^2 for the masses at each radii. For most objects, this involves calculus and is not part of this course. Table 4.1, however, shows the moments of inertia for various shaped objects. In discrete cases, we have the equation

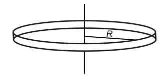
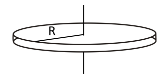
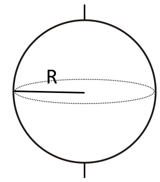
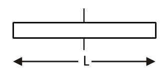
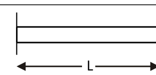
$$I = \sum m_i {r_i}^2. \tag{4.12}$$

More properly, the moment of inertia is given by the volume integral

$$I = \int_0^M r^2 dm,$$

where dm is an infinitesimal mass element and r is the distance from the mass element to the axis of rotation.

Table 4.1: Moments of Inertia for Various Shapes

Object	Axis of Rotation		Moment of Inertia
Thin ring of radius R and mass M	Through center		MR^2
Cylinder of radius R and mass M	Through center		$\frac{1}{2}MR^2$
Sphere of radius R and mass M	Through center		$\frac{2}{5}MR^2$
Long rod of length L and mass M	Through center		$\frac{1}{12}ML^2$
	Around end		$\frac{1}{3}ML^2$

Examples Involving Moments of Inertia

Sections 8-6

Example 1

A disk of radius 15.0 cm and mass 100 g accelerates from rest to an angular speed of 50.0 rpm (revolutions per minute) in 2.50 s. What torque is required to produce this acceleration?

$R = 0.150$ m
$m = 0.100$ kg
$\omega_i = 0$

$$\omega_f = \frac{50.0 \times 2\pi}{60.0\text{s}} = 5.24\,\frac{\text{rad}}{\text{s}}$$
$$\alpha = \frac{\Delta\omega}{t} = \frac{5.24 - 0\,\frac{\text{rad}}{\text{s}}}{2.50\text{s}} = 2.10\,\frac{\text{rad}}{\text{s}^2}$$

For a disc (cylinder)
$I = \frac{1}{2}MR^2 = \frac{1}{2}\,(0.100\text{kg})\,(0.150\text{m})^2 = 1.13 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$
So the torque required is
$\tau = I\alpha = \left(1.13 \times 10^{-3}\ \text{kg}\cdot\text{m}^2\right)\left(2.10\,\frac{\text{rad}}{\text{s}^2}\right) = 2.37 \times 10^{-3}\ \text{N}\cdot\text{m}$

Example 2

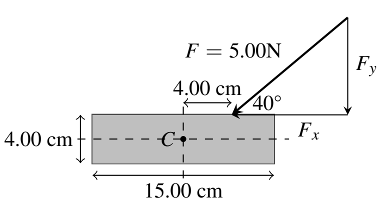


Figure 4.3: Example 2: Torque in both directions provided by a single force

A rod of length 15.00 cm, width 4.00 cm and mass 0.250 kg is rotated around its center by a force of 5.00 N, applied at an angle of 40.0° as shown. How many revolutions has the rod made if this force is applied consistently at this point for 0.300 s (see Figure 4.3)?

It is first necessary to divide the force into x- and y- components (alternatively, we could determine $r_\perp$). The force in the y- direction provides a clockwise rotation (which we will define as the + direction) at a radius of 4.00 cm (r_x) and the force in the x-direction causes a counterclockwise (-) rotation at a radius of 2.00 cm (r_y).

First we calculate F_x and F_y:
$F_x = (5.00\text{ N})(\sin 40.0°) = 3.21\text{ N}$
$F_y = (5.00\text{ N})(\cos 40.0°) = 3.83\text{ N}$

The total torque is $\tau_T = \tau_{CW} - \tau_{CCW}$
$\tau_{CW} = (3.21\text{ N})(0.0400\text{ m}) = 0.128\text{ N}\cdot\text{m}$ and
$\tau_{CCW} = (3.83\text{ N})(0.0200\text{ m}) = 0.0766\text{ N}\cdot\text{m}$

So, $\tau_T = 0.128 - 0.0766\text{ N}\cdot\text{m} = 0.051\text{N}\cdot\text{m}$

For a rod rotating through its center, $I = \frac{1}{12}ML^2$, which in this case is $I = \frac{1}{12}(0.250\text{ kg})(0.1500\text{ m})^2 = 4.69\times10^{-4}\text{ kg}\cdot\text{m}^2$

Angular acceleration can then be calculated:
$\alpha = \frac{\tau}{I} = \frac{0.051\text{N}\cdot\text{m}}{4.69\times10^{-4}\text{ kg}\cdot\text{m}^2} = 1.1\times10^2\ \frac{\text{rads}}{\text{s}}.$

The angle through which the rod has been rotated then can be calculated:
$\theta = \omega_i t + \frac{1}{2}\alpha t^2 = 0 + \frac{1}{2}\left(1.1\times10^2\ \frac{\text{rads}}{\text{s}}\right)(0.300\text{ s}) = 4.9\text{ rad.}$

Since 1 revolution is 2π radians, the number of revolutions is $\frac{4.9}{2\pi}$ or 0.78 revolutions.

Radius of Gyration, Rotational KE and Angular Momentum

Sections 8-7 to 8-8

Radius of Gyration

WHEN DISCUSSING NON-POINT MASSES, such as spheres, rods, discs, etc., it is often convenient to refer to the ***radius of gyration***, k. This is a type of average radius. It is determined by equating Mk^2 with the moment of inertia of an object, where M is the total mass of the object and is analogous to the center of mass. In other words, it behaves the same as if all the mass is at a radius k. For example, the radius of gyration of a sphere can be determined as follows:

$I = Mk^2 = \frac{2}{5}MR^2$ for a sphere of mass M and radius R. So,

$$k = \sqrt{\frac{2}{5}R^2} = \sqrt{\frac{2}{5}}R \approx 0.632R.$$

This means that a sphere of radius 10 m would rotate in the same way as an object of equal mass, whose mass was all distributed 6.32 m away from the axis of rotation (like a ring of radius 6.32 m).

Angular Momentum

JUST AS WITH LINEAR MOMENTUM, angular momentum is conserved. As you would expect, the quantity angular momentum, L, is given by

$$\vec{L} = I\vec{\omega} \tag{4.13}$$

or in linear terms,

$$\vec{L} = m\vec{r} \times \vec{v}$$

where $\times$ represents the vector cross product (which follows a RH rule - the direction of $\vec{r}$ wrapped into the direction of $\vec{v}$, with the thumb giving the direction of the angular momentum).

You will note that angular momentum is a vector, as is angular velocity. To define the direction of both, we consider a right hand rule: To determine the direction of the angular velocity, wrap the fingers of your right hand in the direction of the spinning - the thumb will point in the direction of the angular velocity (and momentum).

As you would expect, angular momentum is also conserved. This explains why a figure skater can speed up spinning by pulling his/her arms in close, or slow down by extending the arms. It also explains why a bicycle is more stable with the wheels spinning. Mathematically, the conservation of angular momentum is written

$$\sum I_i\omega_i = \sum I_i{}'\omega_i{}' \tag{4.14}$$

where I_i and ω_i represent values before an interaction and $I_i{}'$ and $\omega_i{}'$ represent values after that interaction.

Rotational Kinetic Energy

Like linear motion, rotational motion requires energy. Again, as you would expect, the equation parallels the linear one:

$$RE = \frac{1}{2}I\omega^2 \tag{4.15}$$

When considering an object rolling down a incline, or a spinning pulley, part of the energy goes to rotational kinetic energy.

We will look at a few examples involving rotational kinetic energy, including cylinders and spheres rolling downhill, pulleys (without slippage), etc.