Simple Harmonic Oscillators and 2D-Collisions

Simple Harmonic Motion

Sections 11-1 to 11-2

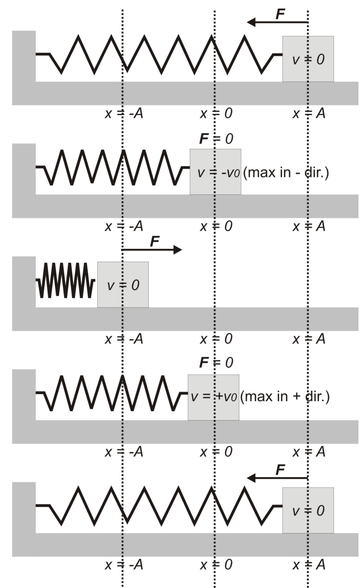


Figure 5.1: Force exerted on and velocity of a mass at different places of an oscillation

PERIODIC MOTION is a motion that repeats itself. You have seen this type of motion in circular motion. You saw it in grade 11 physics with waves. The ***period***, T, is the time for the motion to occur once and the ***frequency***, f is the number of times the motion occurs per unit time (e.g. # of times per second, or Hertz). The relationship between these two is simply

$$f = \frac{1}{T} \tag{5.1}$$

The simplest form of periodic motion is from a mass at the end of a "light" (or massless) spring sliding along a frictionless surface. When a spring is stretched, it applies a restoring force (opposite to the direction of the stretch). As you saw last year, the force is given by

$$\vec{F} = -k\vec{x} \tag{5.2}$$

where k is the spring constant (often in N/m or N/cm) and $\vec{x}$ is the amount and direction of stretch ("−" would represent a compression). The arrows indicate that $\vec{F}$ and $\vec{x}$ are vectors and the negative sign indicates the restorative nature of the force.

Simple harmonic motion (SHM) is exhibited in a system where the restoring force is directly proportional to the negative of the displacement. A system in SHM is known as a ***simple harmonic oscillator (SHO)***. An ideal spring is an excellent example of a SHO

The maximum displacement is called the amplitude, A, of oscillation. When the net force is 0, the oscillator is said to be in the equilibrium position ($x = 0$).

By studying the spring we find that when $x = \pm A$, the oscillator has $v = 0$, but a maximum acceleration (since $F = ma$). But when $x = 0$, the velocity is a maximum (v_0) and $a = 0$, since $F = 0$ (see Figure 5.1).

With no loss of energy, the spring could oscillate back and forth indefinitely.

Energy

THE AMOUNT OF MECHANICAL ENERGY stored in a spring system is simply the sum of the kinetic energy and the potential energy. From last year, we know that the kinetic energy of a moving body is given by $KE = \frac{1}{2}mv^2$. The potential energy in a spring is given by $PE_s = \frac{1}{2}kx^2$ determined using calculus). The energy, therefore is given by

$$E = \frac{1}{2}\left(mv^2 + kx^2\right). \tag{5.3}$$

(Note that for vertical springs, we must also add in a factor which includes the PE due to gravity, but it is otherwise the same.)

When $v = 0$, all of the energy in the system is potential, so $PE_{max} = \frac{1}{2}kA^2$ which is the total energy in the system. So we can determine the maximum velocity, v_0 (pronounced "vee naught"), by setting $x = 0$ and equating the maximum kinetic energy with the maximum potential energy:

$$KE_{max} = \frac{1}{2}m{v_0}^2 = \frac{1}{2}kA^2, \text{ which gives}$$

$$v_0 = \sqrt{\frac{k}{m}}A. \tag{5.4}$$

We can solve for v at any displacement, also using the energy equation. Since the total energy in this system is given by PE_{max},

$$\frac{1}{2}kA^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2.$$

Multiplying by 2 and solving for v^2 gives

$$v^2 = \frac{k}{m}\left(A^2 - x^2\right).$$

Taking the square root gives

$$v = \pm\sqrt{\frac{k}{m}\left(A^2 - x^2\right)}.$$

equation 5.4 gives

$$v = \pm v_0\sqrt{1 - \frac{x^2}{A^2}}\,. \tag{5.5}$$

The Sinusoidal Nature of a SHO (A Mini Calculus Lesson)

Section 11-3

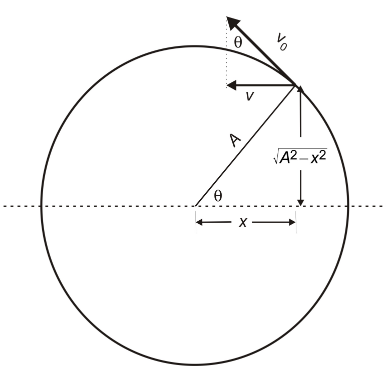


Figure 5.2: Analyzing the $x-$ component of the motion in circular motion for a comparison with SHM.

IF WE COMPARE THE MOTION of a SHO to that of the same mass undergoing circular motion on a horizontal plane, we can determine much about the period and the motion of the SHO.

If we consider the object, of mass m, to be traveling around the circle with speed v_0 (which is equal to the maximum speed of the SHO we talked about last day), and we consider only the x-motion of the mass in circular motion, we see that we can determine a relationship between the speed in the $x-$direction, v, and v_0. By considering similar triangles, we can see that

$$\frac{v}{v_0} = \frac{\sqrt{A^2 - x^2}}{A}$$

or rearranging gives us

$$v = v_0\sqrt{1 - \frac{x^2}{A^2}}$$

and since we are discussing speed instead of velocity, this agrees with equation 5.5, as the direction doesn't matter. Furthermore, if the velocities are at all times equal, it is not a stretch to believe that the SHO will have the same period as the mass in circular motion. You recall

$$T = \frac{2\pi r}{v} \tag{5.6}$$

but in our case, $r = A$ and $v = v_0$ so

$$T = \frac{2\pi A}{v_0}.$$

By substituting for v_0 using equation 5.4 we get

$$T = 2\pi \sqrt{\frac{m}{k}}\ . \tag{5.7}$$

IF WE CONSIDER A SPRING that has been stretched to its full amplitude (i.e., $x_0 = A$) from the reference circle, we can see that the position x can be determined by $x = A\cos\theta$. Since the object goes through 2π radians in a time of one period, we can rewrite this as

$$x = A\cos\frac{2\pi t}{T} \quad \text{or} \quad x = A\cos 2\pi f t. \tag{5.8}$$

The velocity can be determined by using the conservation of energy equation, however, it can be performed quite simply using calculus. We know that velocity it the rate of change of position with respect to time, or

$$v = \frac{\Delta x}{\Delta t}.$$

In other words, it is the slope of the x-t graph. We can't accurately or easily determine the slope by the graph, because the graph is always changing slopes (it is a cosine graph, as shown in your text). But calculus has a means of doing this - it's called differentiation. It turns out that the slope of a cosine function is the negative of the sine function. In calculus, it looks like this

$$v = \frac{dx}{dt} = \frac{d}{dt}\left(A\cos 2\pi f t\right)$$

giving

$$v = -2\pi f A \sin 2\pi f t. \tag{5.9}$$

Notice this is just a bunch of stuff ($-2\pi f A$) times the sine function. We won't worry about how this happens (we'll save that for 1st year university or AP math), but it is important for you to realize that derivatives (or differentiation) are only slopes of graphs.

From equation 5.6 we see that $v = 2\pi f A$ and substituting into equation 5.9 gives

$$v = -v_0 \sin 2\pi f t. \tag{5.10}$$

It is left as an exercise for the reader to show that we get the same result by using equation 5.5 and the trig identity $\sin^2\theta + \cos^2\theta = 1$.

The acceleration is simpler. We could say $a = \frac{dv}{dt}$ (in calculus form) and perform the same feat again, however, we have other relationships which actually simplifies this: $F = -kx$ and $F = ma$. Since we know x as a function of time, we can say

$$a = -\frac{kx}{m} = -\frac{k}{m}\left(A\cos 2\pi f t\right)$$

or

$$a = -a_{max} \cos 2\pi f t \tag{5.11}$$

where $a_{max} = \dfrac{k}{m}A$ is the maximum acceleration that occurs at $x = \pm A$

The Pendulum as an SHO, Damped SHOs and Resonance

Sections 11-4 to 11-6

The Pendulum

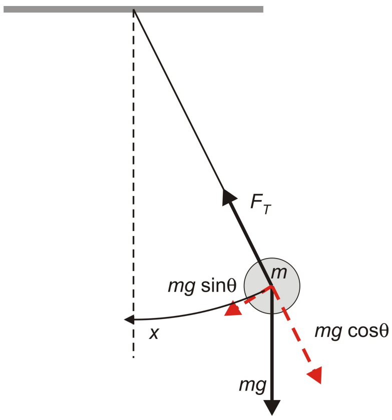


Figure 5.3: Simple Pendulum

IF WE APPLY OUR KNOWLEDGE of forces to the pendulum, we recognize that the forces on the bob are caused by the tension in the string (F_T) and gravity (mg). By dividing the gravitational force into components, we can see that at any point

$$F_T = mg\cos\theta,$$

since the string is not getting longer or shorter, and the restoring force is the component of the force perpendicular to the tension

$$F_x = -mg\sin\theta$$

(the negative sign indicates a direction opposite the swing of the pendulum).

This is not truly a SHO if we return to our definition of SHM. The restoring force must be proportional to the displacement. However, for small angles, the sine of the angle is *approximately* equal to the angle, i.e., $\sin\theta \approx \theta$. [3]

[3] This comes from the Taylor's series expansion of $\sin\theta$ which says $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots$

We will define our x−axis to be along the circle marked out by the pendulum with the origin at the bottom of the swing (i.e., $x = 0$). We know from our study of rotational motion that $x = \theta L$, where θ is in radians and L is the length of the pendulum. Using the small angle approximation

$$F_x = -mg\sin\theta \approx -\frac{mgx}{L}.$$

So for small angles, a pendulum approximates a simple harmonic oscillator. We say that the ***effective force constant***, k, is the proportionality constant, so in this case

$$k = \frac{mg}{L}.$$

We can then substitute this into equation 5.7, for the period of a SHO and we get

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{mL}{mg}}$$

giving

$$T = 2\pi\sqrt{\frac{L}{g}}\,. \tag{5.12}$$

This shows that the period of the pendulum is dependent only on the length of the pendulum and the acceleration of gravity, and that (unlike springs) the period is independent of the mass.

The Damped Harmonic Oscillator

In reality, we do not have systems that can maintain a constant amplitude over time. Friction (either air resistance, within the spring at the molecular level, or other similar forms) gradually reduces the amount of energy in the system, thereby decreasing the amplitude. You know this from common practice: If you start a pendulum swinging, it eventually dies down. Cuckoo or grandfather clocks are wound by pulling up masses, old watches by winding springs, thus adding *PE* to the system.

Many systems, left to themselves, will decay exponentially. Often damped harmonic oscillators are described by two values: the frequency, f, and the time for the amplitude to decrease to $\frac{1}{e}$ its original value[4], t_e. The product of these two values is known as the Q-value (or quality value) of an oscillator, i.e., $Q = 2\pi f t_e$. An oscillator which has a high Q-value will undergo much less energy loss that an oscillator with a low Q-value.

[4] e is an irrational constant, much like π. It's value is approximately $e = 2.718281828459045235360287471352 7\ldots$

Forced Vibrations: Resonance

Most objects, such as a pendulum or a spring, have a natural frequency at which they like to oscillate. The frequency at which a pendulum oscillates is determined by the length of the bob. If you "drive" the system at the same frequency, you can increase the amplitude. For example, on a swing, if you move your legs forward and backward at the same frequency as the swing, you will increase the amplitude of your swing. This is known as resonance.

If, however, you drive the system at a different frequency, the oscillator will vibrate at the frequency caused by the external force, and we have a forced vibration. The amplitude of this oscillation depends upon the difference between the natural frequency, f_0, and the forced frequency, f.

Conservation of Momentum in 2 Dimensions

Sections 7-1 to 7-7

Momentum and Impulse

Momentum is simply the product of an object's mass and its velocity. As such, it is a vector quantity, and is therefore directional. In vector form

$$\vec{p} = m\vec{v}. \tag{5.13}$$

In reality, momentum is the quantity which Newton refers to in his second law of motion. You recall from grade 11 that

$$\vec{J} = \vec{F}\Delta t = \Delta\vec{p}. \tag{5.14}$$

Newton's second law ($\sum \vec{F} = m\vec{a}$) is actually expressed in terms of momentum. In calculus form it is just

$$\sum \vec{F} = \frac{d\vec{p}}{dt}$$

or the rate of change of momentum with respect to time. If we substitute Δ for the d and rearrange, we see that we get the momentum-impulse relation (equation 5.14). This is a more exact form of Newton's second law because it is possible for the mass to be changing as well as the velocity. In fact this is exactly the case with a rocket is space. It is literally expelling fuel out the back which causes a change in the velocity of the rocket.

The quantity J in equation 5.14 is known as ***impulse***. Note that the units of momentum and impulse are the same and they can be represented in either $\frac{\text{kg} \cdot \text{m}}{\text{s}}$ or $\text{N} \cdot \text{m}$.

Conservation of Momentum

We can show that in a collision between two bodies (and therefore between any number of bodies) that the total momentum before the collision is equal to the total momentum after the collision. This statement is known as the ***conservation of momentum***.

Let us assume that there are two bodies with masses m_1 and m_2, and velocities $\vec{v}_1$ and $\vec{v}_2$ before a collision. We will assume the masses remain unchanged (although that is not necessary to this proof) and the velocities after the collision are $\vec{v}_1'$ and $\vec{v}_2'$ respectively. Then

$$\Delta \vec{p}_1 = m_1 \vec{v}_1' - m_1 \vec{v}_1$$

and

$$\Delta \vec{p}_2 = m_2 \vec{v}_2' - m_2 \vec{v}_2.$$

During the collision, object one exerts a force $\vec{F}_{12}$ on object two and object two exerts a force $\vec{F}_{21}$ on object one, both for the time, t, during which they are in contact. By Newton's third law, we know that $\vec{F}_{12} = \vec{F}_{21}$, that is the two forces are equal and opposite.

But by Newton's second law, $\Delta \vec{p}_1 = \vec{F}_{12} t$ and $\Delta \vec{p}_2 = \vec{F}_{21} t$, so $\Delta \vec{p}_1 = -\Delta \vec{p}_2$. (Note this means the two impulses are equal and opposite). This means that

$$m_1 \vec{v}_1' - m_1 \vec{v}_1 = m_2 \vec{v}_2 - m_2 \vec{v}_2'.$$

Grouping the before the collision quantities and the after collision quantities, we get

$$m_1 \vec{v}_1 + m_2 \vec{v}_2 = m_1 \vec{v}_1' + m_2 \vec{v}_2'$$

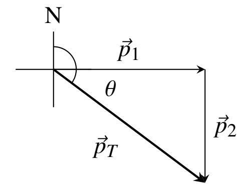
or in general for n bodies

$$\sum_{i=1}^{n} \vec{p}_i = \sum_{i=1}^{n} \vec{p}_i' \tag{5.15}$$

where $\vec{p}_i$ and $\vec{p}_i'$ represent the momentum of the i^{th} body before and after the collision.

Conservation of Momentum Example

Two balls, of mass 2.0 kg and 3.0 kg respectively collide. Before the collision, the 2.0 kg ball has a velocity of 4.0 m/s [E] and the 3.0 kg ball has a velocity of 2.0 m/s [S]. The two balls stick together during the collision. What is the combined velocity of the balls after the collision?

Before Collision	After Collision
$\vec{p}_1 = m_1\vec{v}_1 = (2.0\text{ kg})\left(4.0\ \frac{\text{m}}{\text{s}}\ [E]\right) = 8.0\frac{\text{kg·m}}{\text{s}}\ [E]$ $\vec{p}_2 = m_2\vec{v}_2 = (3.0\text{ kg})\left(2.0\ \frac{\text{m}}{\text{s}}\ [E]\right) = 6.0\frac{\text{kg·m}}{\text{s}}\ [S]$ So $p_T = \sqrt{{p_1}^2 + {p_2}^2} = 10\frac{\text{kg·m}}{\text{s}}$ and $\theta = \tan^{-1}\frac{6.0\ \frac{\text{kg·m}}{\text{s}}}{8.0\ \frac{\text{kg·m}}{\text{s}}} = 37°$ So $\vec{p}_T = 10\frac{\text{kg·m}}{\text{s}}\ @127°$	$\vec{v} = \frac{\vec{p}_T}{m_T} = 2.0\frac{\text{m}}{\text{s}}\ @127°$ since the two masses are joined together. 

Elastic vs. Inelastic Collisions

Sections 7-1 to 7-7

Energy and Collisions

WE CAN CATEGORIZE COLLISIONS as either elastic or inelastic. In an elastic collision, the total kinetic energy after the collision is equal to the total kinetic energy before the collision. The collisions are called elastic, because for a brief period of time, some or all of the kinetic energy is stored momentarily in the form of elastic potential energy (the ability of the object to "spring back"). The collision of two hard balls (billiard balls or curling stones) is often used as an example of an elastic collision.

In the case of elastic collisions, besides using the conservation of momentum relation, we can also use

$$\sum KE_{before} = \sum KE_{after}$$

It should go without saying (but we'll say it anyway) that an inelastic collision is a case where the kinetic energy after the collision is not equal to the kinetic energy before the collision. In ordinary collisions, it is not possible for the kinetic energy to increase, unless energy has been released in the collision (such as a spring or an explosion).

We will look at an example of an elastic collision and its solution.

Angular Momentum

IT SHOULD BE NOTED that angular momentum is conserved in the same way linear momentum is. This is a partial explanation of why it is more difficult to tip a moving bicycle than a stationary one. We saw examples of this in Chapter 4 (see page 38).